Options pricing using Fourier transform

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"Mathematical analysis is a coherent symphony of the infinite."
(David Hilbert)

1 Introduction

To get around the lognormal assumption for asset prices and its consequences, we can use other methods. Rather than calculating the discounted expectation of the final payoff directly, it may be more convenient to use the Fourier transform and the characteristic function of the Lévy process, which is easier to manipulate than the density function itself.

So, by using Fast Fourier Transform (FFT) algorithms, we can carry out the evaluation of options in an efficient way. We therefore incorporate a more general and realistic structure of asset returns, such as excess kurtosis and stochastic volatility.

In this paper, we highlight Carr and Madan's fast Fourier transform approach to valuing European call options. We restrict ourselves to applications in the context of continuous diffusion processes. Another article will deal with the case of jump processes (Merton and variance-gamma).

2 Fourier transform

Let $f \in L^1(\mathbb{R}^d)$. The Fourier transform of f, denoted $\hat{f}$, is defined on $\mathbb{R}^d$ by:

$$\forall \xi \in \mathbb{R}^d, \quad \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi}\, dx. \tag{1}$$

This function is linear and well defined since, $\forall f \in L^1(\mathbb{R}^d), \forall \xi \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |f(x) e^{-ix\xi}|\, dx = \int_{\mathbb{R}^d} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R}^d)} < \infty. \tag{2}$$

For X a random variable with density f_X with respect to the Lebesgue measure on $\mathbb{R}^d$, its characteristic function is defined as: $\phi_X : t \in \mathbb{R} \mapsto \mathbb{E}(e^{itX})$. Using the transfer theorem, we obtain:

$$\phi_X(t) = \int_{\mathbb{R}^d} e^{itx} f(x)\, dx = \hat{f}(-t). \tag{3}$$

Example 1 *(Fourier transform of indicator function): let $a, b \in \mathbb{R}$ and $f = 1_{[a,b]}$. The Fourier transform of f is:*

$$\hat{f}(\xi) = \int_a^b e^{-ix\xi}\, dx = \begin{cases} \dfrac{2e^{-i\frac{a+b}{2}\xi}\sin\left(\frac{b-a}{2}\xi\right)}{\xi}, & \text{if } \xi \neq 0 \\ b-a, & \text{if } \xi = 0 \end{cases}$$

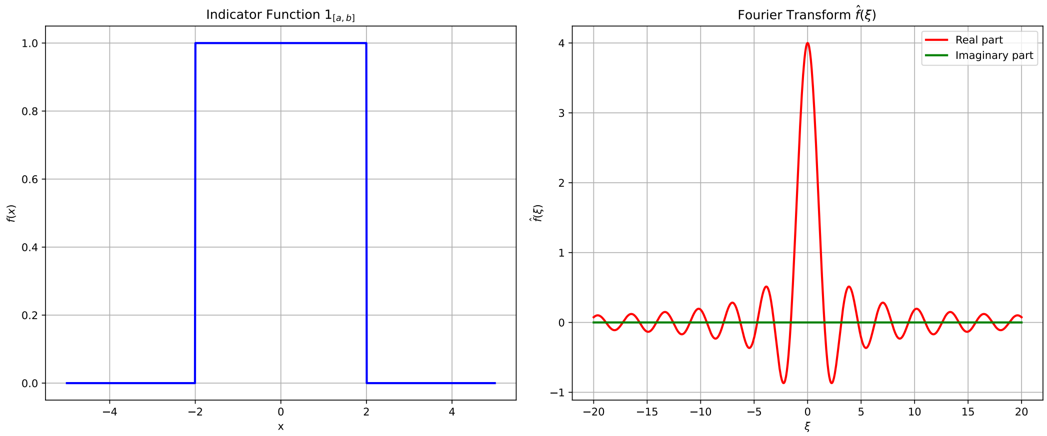


Figure 1: $f = 1_{[-2,2]}$ and its Fourier transform

Example 2 *(Cauchy distribution): let X be a random variable following a Cauchy distribution with parameter 1 i.e $X \sim \mathfrak{C}au(1)$. For $\xi \in \mathbb{R}$, we have:*

$$\phi_X(\xi) = \mathbb{E}(e^{iX\xi}) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{ix\xi}}{1+x^2}\, dx = \frac{1}{\pi}\hat{f}(-\xi) = e^{-|\xi|}.$$

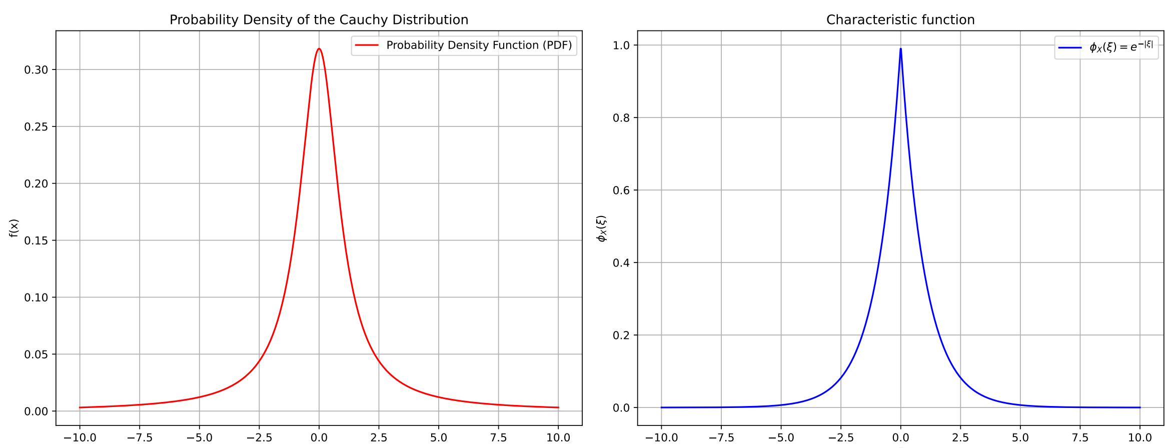


Figure 2: Density and characteristic function for $X \sim \mathfrak{C}\mathbf{au}(1)$

Example 3 *Let $f(x,y) = e^{-(x^2+y^2)}$. The Fourier transform is:*

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}^2} f(x,y) e^{-i(x\xi + y\eta)}\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} e^{-i(x\xi+y\eta)}\, dx\, dy$$

i.e:

$$\hat{f}(\xi,\eta) = \left(\int_{-\infty}^{\infty} e^{-x^2} e^{-ix\xi}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} e^{-iy\eta}\,dy\right)$$

with:

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-ix\xi}\,dx = e^{-\frac{\xi^2}{4}}\sqrt{\pi}$$

we have:

$$\hat{f}(\xi,\eta) = \left(\sqrt{\pi}e^{-\frac{\xi^2}{4}}\right)\left(\sqrt{\pi}e^{-\frac{\eta^2}{4}}\right) = \pi e^{-\frac{\xi^2+\eta^2}{4}}$$

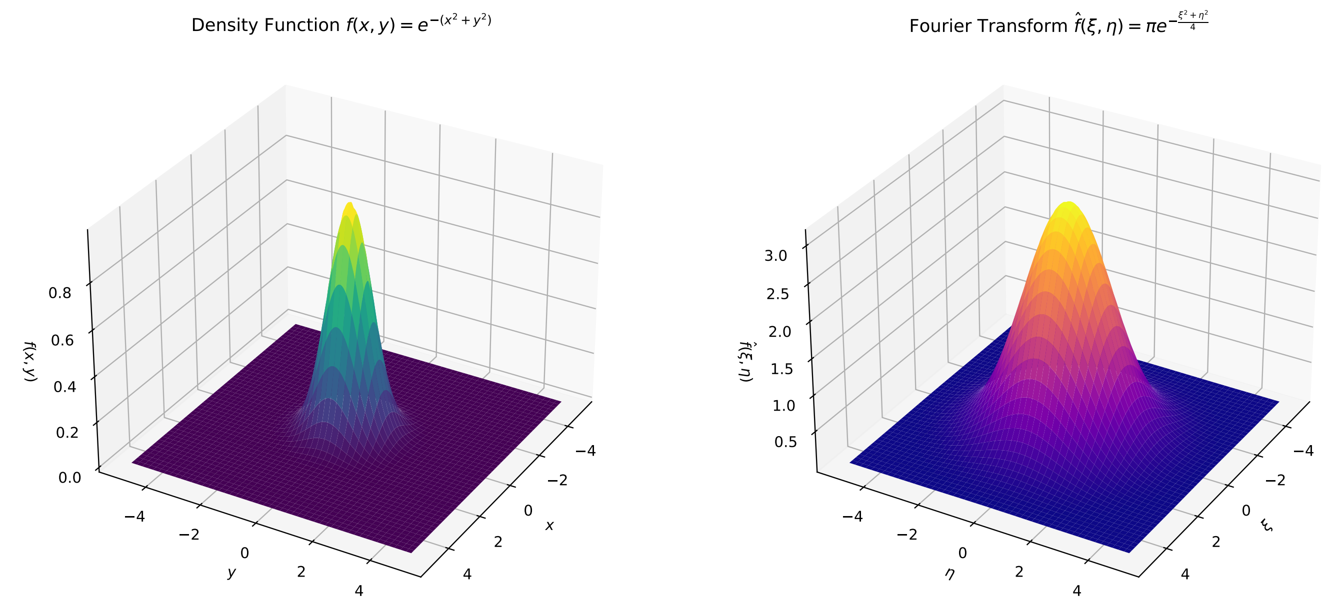


Figure 3: Density and fourier transform for f

3 Properties

3.1 Translation:

let $f \in L^1(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$. Then,

$$\widehat{\tau_a f} = \xi \mapsto e^{-ia\xi}\hat{f}(\xi).$$

Proof: For $\xi \in \mathbb{R}^d$,

$$\widehat{\tau_a f}(\xi) = \int_{\mathbb{R}^d} f(x-a)e^{-ix\xi}\,dx = \int_{\mathbb{R}^d} f(u)e^{-i(u+a)\xi}\,du = e^{-ia\xi}\hat{f}(\xi).$$

□

3.2 Continuity:

$\forall f \in L^1(\mathbb{R}^d)$, $\hat{f}$ is uniformly continuous on $\mathbb{R}^d$. So that it's continuous.

3.3 Fourier operator on $L^1(\mathbb{R}^d)$):

Consider the Fourier operator on $L^1(\mathbb{R}^d)$ defined by:

$$\begin{array}{cccc} F_1 : (L^1(\mathbb{R}^d), \|\cdot\|_{L^1(\mathbb{R}^d)}) & \to & (C_0^0(\mathbb{R}^d,\mathbb{R}), \|\cdot\|_\infty) \\ f & \mapsto & \hat{f} \end{array}$$

then, F_1 is a continuous linear operator with subordinate norm equal to 1.

Proof: It is clearly a linear operator and $|||F_1|||_{Lc(L^1,C_0^0)} \leq 1$. We know that:

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \frac{dx}{1+x^2} = \pi.$$

Moreover,

$$\|\xi \mapsto \pi e^{-|\xi|}\|_{L^\infty(\mathbb{R})} = \pi.$$

□

3.4 Convolution:

Let $f, g \in L^1(\mathbb{R}^d)$ be two functions. Then, they are convolvable, i.e., for almost every $x \in \mathbb{R}^d$,

$$y \in \mathbb{R}^d \mapsto f(y)g(x-y) \in L^1(\mathbb{R}^d).$$

Moreover, $f * g \in L^1(\mathbb{R}^d)$ and we have the following relation:

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Remarque 1 *If X and Y are two independent random variables with densities, then the density of $X+Y$ is given as $f_X * f_Y$. Thus,*

$$\phi_{X+Y} = \widehat{f_{X+Y}}(-\cdot) = \widehat{f_X * f_Y}(-\cdot) = \widehat{f_X}(-\cdot) \cdot \widehat{f_Y}(-\cdot) = \phi_X \phi_Y.$$

3.5 Duality:

Let $f, g \in L^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} f(x)\widehat{g}(x)\,dx = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,dx.$$

3.6 Fourier transform of the Derivative:

Let $f \in L^1(\mathbb{R}^d)$, and let $j \in \{1,\dots,d\}$ such that $\partial_{x_j} f$ exists and is in $L^1(\mathbb{R}^d)$. Then, we have the following relation:

$$\forall \xi \in \mathbb{R}^d, \quad \widehat{\partial_{x_j} f}(\xi) = i\xi_j \widehat{f}(\xi).$$

3.7 Derivative of the Fourier transform:

Let $f \in L^1(\mathbb{R}^d)$. Suppose that for $j \in \{1,\dots,d\}$, $x \mapsto x_j f(x) \in L^1(\mathbb{R}^d)$. Then, $\widehat{f}$ has a partial derivative with respect to e_j, and the following relation holds:

$$\partial_{\xi_j} \widehat{f}(\xi) = -i\, \widehat{x_j f(x)}(\xi).$$

4 Consequences

4.1 Fourier transform of f_a:

Let $a > 0$, we define: $f_a : x \in \mathbb{R} \mapsto e^{-\frac{ax^2}{2}}$. Then, we have:

$$\forall \xi \in \mathbb{R}, \quad \widehat{f_a}(\xi) = \sqrt{\frac{2\pi}{a}}\, e^{-\frac{\xi^2}{2a}}.$$

4.2 Fourier inversion in $L^1(\mathbb{R}^d)$:

Let $f \in L^1(\mathbb{R}^d)$ such that $\hat{f} \in L^1(\mathbb{R}^d)$. Then,

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi, \quad \text{almost everywhere in } x.$$

4.3 Injectivity of the Fourier operator:

The Fourier operator F_1 is injective.

Proof: Let $f \in L^1(\mathbb{R}^d)$ such that $F_1(f) = 0$. Then, $\hat{f} = F_1(f) = 0 \in L^1(\mathbb{R}^d)$. By Fourier inversion, we have:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi = 0, \text{ almost everywhere in } x \in \mathbb{R}^d.$$

□

4.4 Non-surjectivity of the Fourier operator:

The Fourier operator F_1 is not surjective.

Remarque 2 *The only solution to the equation $f * f = f$ in $L^1(\mathbb{R}^d)$ is the zero function.*

4.5 About $L^2(\mathbb{R}^d)$

To define the Fourier transform on $L^2(\mathbb{R}^d)$, we can't use the definition given for $L^1(\mathbb{R}^d)$. For this, we will focus on the properties of the Fourier transform on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and then extend the result by density.

5 Lévy processes

An adapted real-valued stochastic process X_t, with $X_0 = 0$, is called a Lévy process if it observes the following properties:

- Independent increments: for every increasing sequence of times $t_0, t_1, \ldots, t_n$, the random variables, $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
- Time-homogeneous: the distribution of $\{X_{t+s} - X_s : t \geq 0\}$ does not depend on s.
- Stochastically continuous: for any $\epsilon > 0$, $P[|X_{t+h} - X_t| > \epsilon] \to 0$ as $h \to 0$.
- Cadlag process: it is right continuous with left limits as a function of t.

Lévy processes are a combination of a linear drift, a Brownian process, and a jump process. When the Lévy process X_t jumps, its jump magnitude is non-zero. The Lévy measure w of X_t defined on $\mathbb{R} \setminus \{0\}$ dictates how the jump occurs. In the finite-activity models, we have:

$$\int_{\mathbb{R}} w(dx) < 1.$$

In the infinite-activity models, we observe:

$$\int_{\mathbb{R}} w(dx) = 1.$$

And the Poisson intensity cannot be defined. Loosely speaking, the Lévy measure $w(dx)$ gives the arrival rate of jumps of size $(x, x+dx)$.

The characteristic function of a Lévy process can be described by the Lévy-Khinchine representation:

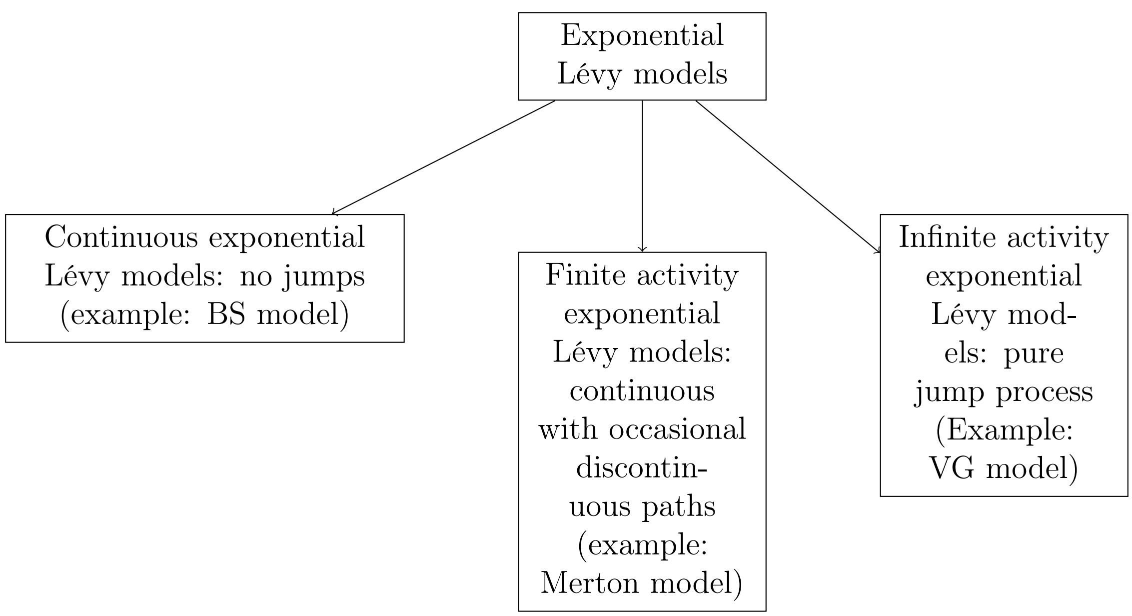
$$\phi(u) = E[e^{iuX_t}] = \exp\left(iau - \frac{1}{2}\sigma^2 t u^2 + t\int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux1_{|x|<1}\right) w(dx) \right),$$

where:

$$\int_{\mathbb{R}} \min(1, x^2) w(dx) < \infty, \quad a \in \mathbb{R}, \quad \sigma^2 \geq 0.$$

We identify a as the drift rate and σ as the volatility of the diffusion process. Here, $\phi(u)$ is called the characteristic exponent of X_t. Actually, X_t has the same distribution as tX_1. All moments of X_t can be derived from the characteristic function since it generalizes the moment-generating function to the complex domain. Indeed, a Lévy process X_t is fully specified by its characteristic function ϕ.

Here is a simplified diagram to give you an overview:



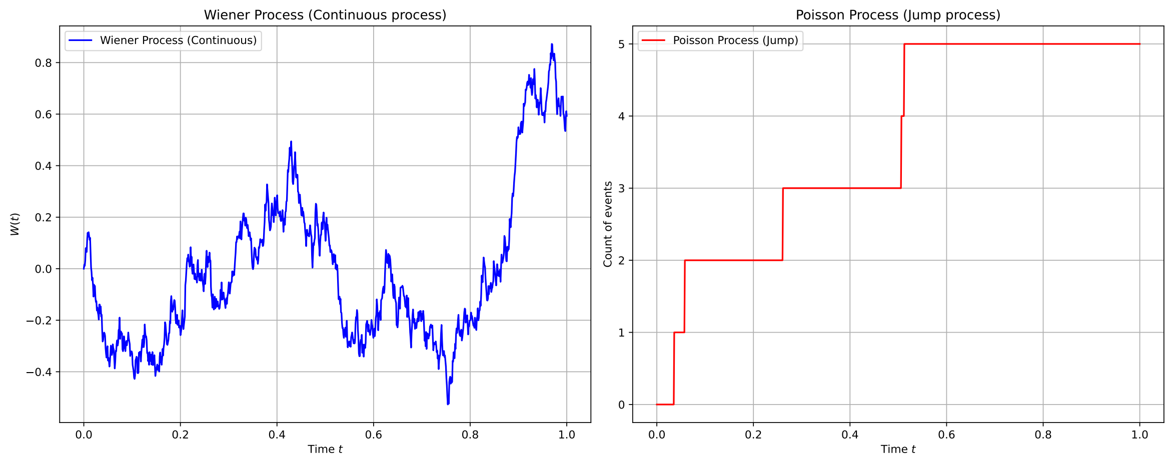


Figure 4: Lévy processes

6 Geometric Brownian Motion (GBM)

Let the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ on which is defined the process of prices $(S_t)_{t\in[0,T]}$ and a risk-neutral probability $\mathbb{Q}$ equivalent to $\mathbb{P}$. Since there is no arbitrage opportunity and the market is complete, this risk-neutral probability exists.

The dynamics of underlying asset under the risk-neutral measure $\mathbb{Q}$ are given by the stochastic differential equation (SDE):

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}, \tag{4}$$

where:

- S_t: price of the asset at time t.
- r: risk-free rate, σ: volatility.
- $dW_t^{\mathbb{Q}}$: increment of a Wiener process.

We therefore know that the solution (derived using Itô's lemma) is given by:

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}\right). \tag{5}$$

where S_0 is the initial price of the asset at $t = 0$.

In fact, $S_t \sim \text{Log}\mathcal{N}(\eta, \zeta^2)$, i.e $X_t = \log(S_t) \sim \mathcal{N}(\eta, \zeta^2)$. and $S_t = e^{X_t}$

With: $\eta = \log(S_0) + (r - \frac{\sigma^2}{2})t$ and $\zeta^2 = \sigma^2 t$.

The payoff of a European call option with strike price K and maturity T is given by:

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right]. \tag{6}$$

where $(x)^+ = \max(x, 0)$.

Using the information available up to valuation date t, we have:

$$C_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)|\mathcal{F}_t^+\right]. \tag{7}$$

Remarque 3 *(About the Radon-Nikodym Theorem) The Radon-Nikodym theorem provides a way to change from the real-world probability measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$. We have the following relationship:*

$$L_T = \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \textit{and} \quad \forall t \in [0, T],\ L_t = \mathbb{E}^{\mathbb{P}}[L_T|\mathcal{F}_t].$$

In particular, if $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$, let $\lambda = \frac{\mu - r}{\sigma}$, with λ: market price of risk. We define $W_t = B_t + \lambda t$, $t \in [0, T]$, thanks to Girsanov, the process $(W_t)_{t\in[0,T]}$ is brownian motion under $\mathbb{Q}$ such as:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\lambda B_T - \frac{1}{2}\lambda^2 T} \tag{8}$$

The price of the European call (with no dividend) option is:

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2). \tag{9}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{10}$$

$$d_2 = d_1 - \sigma\sqrt{T}, \tag{11}$$

and $N(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution.

Remarque 4 *in the event of a dividend q, we have, under $\mathbb{P}$, the SDE:*

$$dS_t = (\mu - q)S_t dt + \sigma S_t dB_t. \tag{12}$$

Then under $\mathbb{Q}$, the price of the European call (with dividend) option is:

$$C = S_0 e^{-qT}N(\tilde{d}_1) - Ke^{-rT}N(\tilde{d}_2). \tag{13}$$

where:

$$\tilde{d}_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{14}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T}, \tag{15}$$

Remarque 5 *(Put option price via Put-Call parity)*

The put-call parity relationship for European options is:

$$C - P = S_0 - Ke^{-rT}. \tag{16}$$

From this relationship, we can find the price of the put (with no dividend) option as follows:

$$P = Ke^{-rT}N(-d_2) - S_0N(-d_1) \tag{17}$$

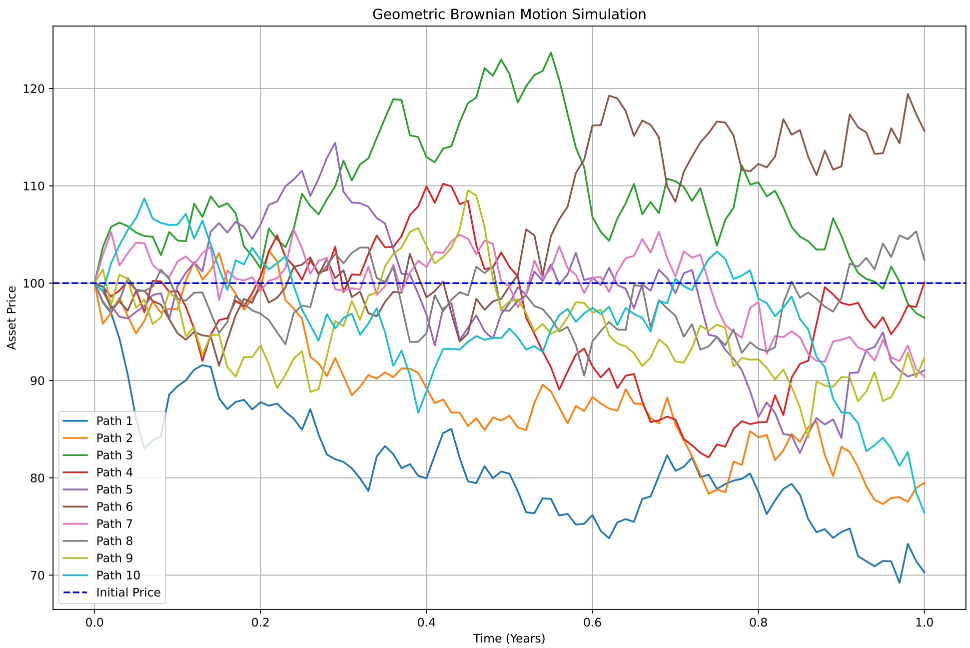


Figure 5: GBM paths

7 Option Call price and Fourier inversion

Let X random variable and f its density. the characteristic function

$$\phi_X(\xi) = \mathbb{E}\big[e^{i\xi X}\big] = \int_{\mathbb{R}} e^{i\xi x} f_X(x)dx = \hat{f}_X(-\xi)$$

exists because $|e^{i\xi x}| = 1$, $\int_{\mathbb{R}} |f_X(x)|dx = 1$ (f is a density of probability), and:

$\left|\int_{\mathbb{R}} e^{i\xi x} f_X(x)dx\right| \leq \int_{\mathbb{R}} |e^{i\xi x} f_X(x)|dx < \infty.$

The inverse Fourier transform is given by:

$$f_X(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}\phi_X(\xi)d\xi$$

.

By inversion theorem, we have:

$$F_X(x) = \frac{1}{2} - \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}\phi_X(\xi)\frac{1}{i\xi}\,d\xi$$

and by Gil Pelaez formulas,

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re e\left[\frac{e^{-i\xi x}\phi_X(\xi)}{i\xi}\right]d\xi$$

. Then the density is:

$$f_X(x) = \frac{1}{\pi}\int_0^\infty \Re e\left[e^{-i\xi x}\phi_X(\xi)\right]d\xi$$

.

In fact, the initial price of a call option can be written as:

$$C = S_0\Pi_1 - Ke^{-rT}\Pi_2, \tag{18}$$

where Π_1, Π_2 are the option's delta and the risk-neutral probability of finishing in the money respectively. For example (BS model) we have: $\Pi_1 = N(d_1)$ and $\Pi_2 = N(d_2)$.

Now we deduce Π_1 and Π_2 using the Gil Pelaez formula like this:

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re e\left[\frac{e^{-i\xi\kappa}\phi_X(\xi)}{i\xi}\right]d\xi, \tag{19}$$

and

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re e\left[\frac{e^{-i\xi\kappa}\phi_X(\xi - i)}{i\xi\phi_X(-i)}\right]d\xi. \tag{20}$$

With $\kappa = \log(\frac{K}{S_t})$ and $S_T = S_t\exp(X_T)$.

Example 4 *:*

```
# Initialization parameters
S0 = 55.0   # Spot stock price
K = 50.0    # Strike price
r = 0.04    # Risk-free rate
T = 1.0     # Maturity in years
std = 0.3   # Volatility
```

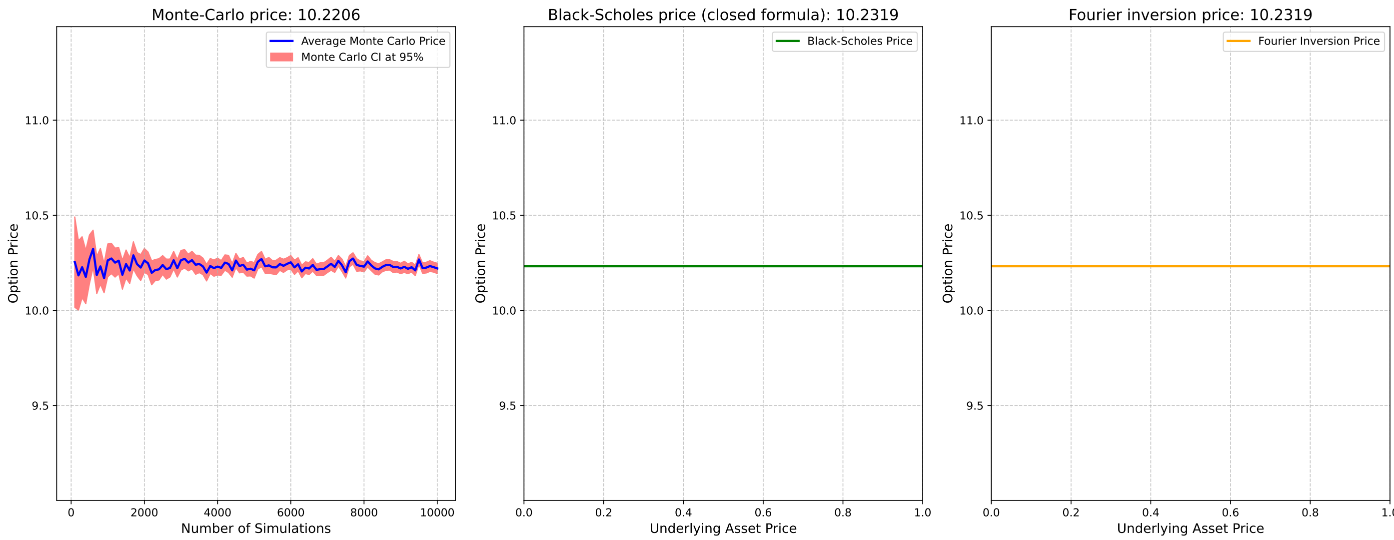


Figure 6: European call option price using 3 methods

Summary of Option Pricing Methods

Method	**Execution Time (s)**
Monte Carlo	0.08095145225524902
Black-Scholes	0.06067371368408203
Fourier Inversion	0.018828630447387695

8 Fast Fourier transform (FFT)

The FFT is a numerical algorithm used to compute the Discrete Fourier Transform (DFT) of a sequence of data efficiently.

The DFT applies to a discrete deterministic signal, by transforming this signal into a series of frequency components corresponding to the amplitudes and phases of the different frequencies present in the signal. This is method that transforms a sequence of N points into a series of frequency components. DFT allows transforming from the time domain (or spatial domain) to the frequency domain, which is useful for analyzing the frequencies present in a signal or dataset.The DFT is expressed by the following formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-i\frac{2\pi}{N}kn}, \quad k = 0, 1, \ldots, N-1. \tag{21}$$

where:

- $x(n)$ represents the input values of the sequence (time or space, for instance).
- $X(k)$ represents the output values (the frequency components).
- N is the total number of points in the series.
- $e^{-i\frac{2\pi}{N}kn}$ is the complex exponential encoding the frequency.

Remarque 6 ***(Computational Complexity)***

The DFT has a calculation complexity of $O(N^2)$, because it requires each point in the signal to be multiplied by a complex exponential coefficient for each frequency.

The FFT, on the other hand, is an algorithm that exploits the symmetries and properties of complex exponentials to calculate the DFT much more quickly, with a complexity of $O(NlogN)$. This reduction in complexity is particularly useful.

9 Carr-Madan approach

The method solves the singularity problem at $\xi = 0$ in the Fourier inversion method. The characteristic function of the probability density of the price of a risk-neutral asset is known analytically. From this function, we can quickly calculate option prices using the Fourier transformation via the FFT.

Let $\kappa = \log K$, $s_t = \log S_t$, and Φ_T be the characteristic function of the terminal log-asset price s_T.

We define a modified call price:

$$c_T(\kappa) = e^{\alpha\kappa} C_T(\kappa). \tag{22}$$

With $C_T(\kappa)$ the initial price of the call option with maturity T and log-strike κ. For a well-chosen α, $e^{\alpha\kappa}$ solves singularity problems.

If ϵ_T is the Fourier transform of $c_T(\kappa)$, we have:

$$\epsilon_T(\xi) = \frac{e^{-rT}\phi_T\left(\xi - (\alpha+1)i\right)}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi}. \tag{23}$$

Then, we apply the inverse Fourier transform to obtain:

$$C_T(\kappa) = \frac{e^{-\alpha\kappa}}{\pi} \int_0^{+\infty} e^{-i\xi\kappa} \epsilon_T(\xi) d\xi. \tag{24}$$

But then we use the fast Fourier transform algorithm in the quadrature of the integral, if γ is the spacing size, we determine the uniform partition of N points like $\xi_j = \gamma j$, for $j = 0, 1, ...N-1$.
We truncate the improper integral at $N\gamma$. Then we use Simpson's rule to increase precision.

Let $b = \frac{N\lambda}{2}$, $\lambda\gamma = \frac{2\pi}{N}$, $\kappa_m = -b + \lambda m$, for $m = 0, 1, ..., N-1$, application of the FFT leads to:

$$C(\kappa_m) \approx \frac{e^{-\alpha\kappa_m}}{\pi} \sum_{j=0}^{N-1} e^{\frac{-2\pi i j m}{N}} e^{ib\xi_j} \epsilon_T(\xi_j) \frac{\gamma}{3} [3 + (-1)^{j+1} - \delta_j]. \tag{25}$$

with the Kronecker symbol:

$$\delta_j = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's restrict ourselves to the continuous market model and let's apply the FFT algorithm to the BSM model.

Example 5 *: Assume the following parameters:*

```
# Initialization parameters
S0 = 100.00  # Spot stock price
K = 100.00   # Strike price
r = 0.05   # Risk-free rate
T = 1.0    # Maturity in years
std = 0.2  # Volatility
```

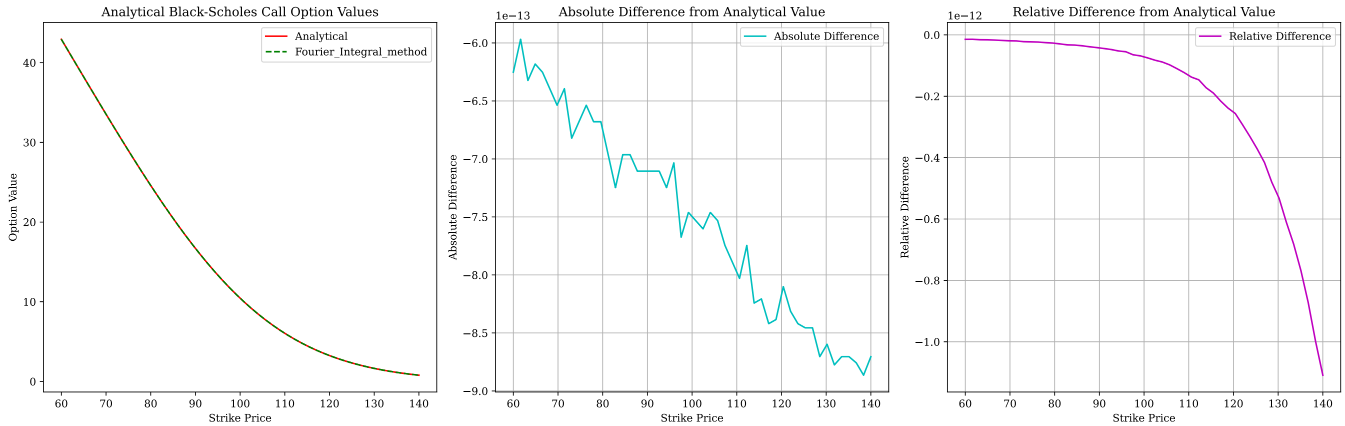


Figure 7: Analytical Black-Scholes vs Fourier integral method

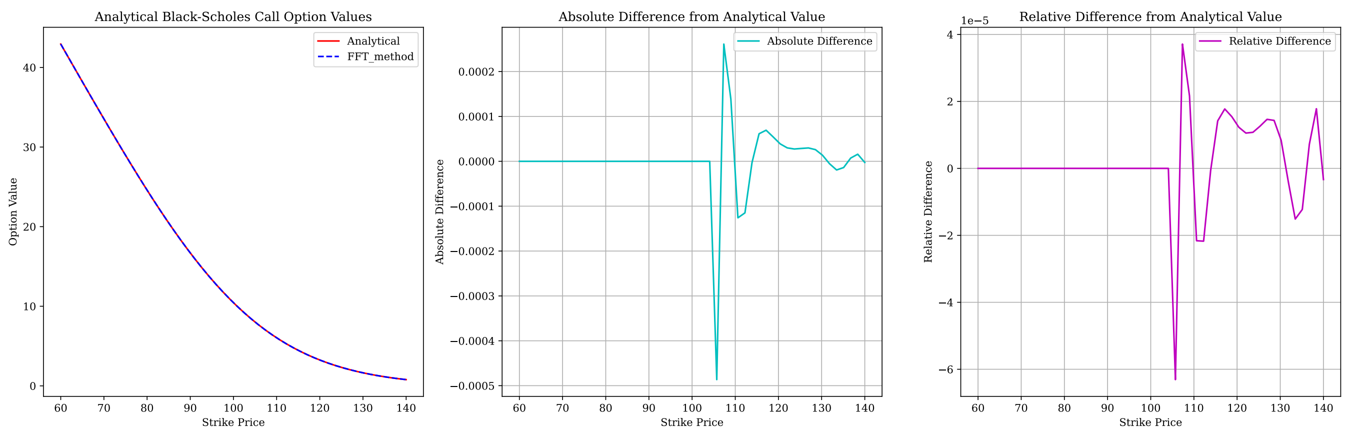


Figure 8: Analytical Black-Scholes vs FFT method

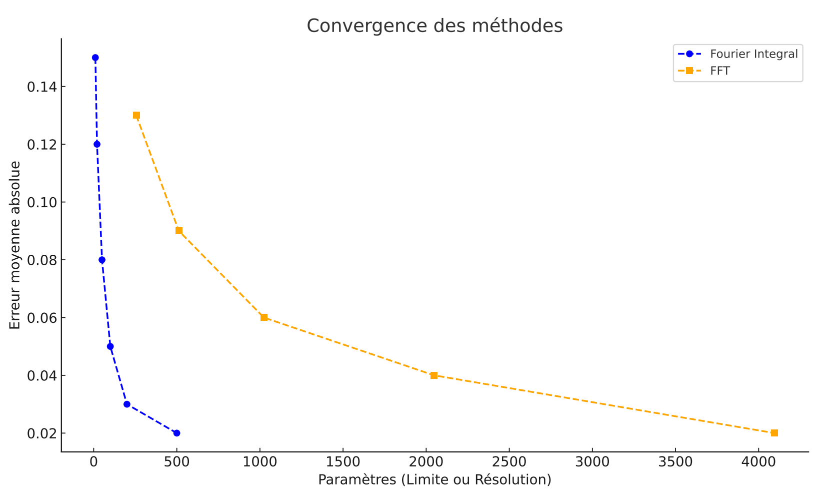


Figure 9: Convergence comparison

For a given accuracy, the integral Fourier method achieves a low error with reasonable integration limits. The FFT method is useful for fast calculations over large ranges, but can be less accurate for modest sample sizes.

To closely resemble how options are listed in financial markets, where K is given and $S0$ is observed, we use a range of strike prices to visualize how the option value changes. So, for a given spot price $S0$, the higher the strike price K, the less valuable the option becomes.

Option pricing based on the Fourier method combines the generality of the risk-neutral pricing approach with the convenience of a closed pricing formula as in the BSM configuration. The speed and precision of these methods are unequivocal.

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